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ANALYSIS OF NON-LINEAR DYNAMIC SYSTEMS EXCITED BY INTENSE NON-WHITE NOISE†

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A method of finding the stationary moments of the solution of non-linear stochastic equations with additive Gaussian random action is proposed, based on the use of matrix continued fractions. The method imposes no *a priori* limitations on the intensity and correlation time of the noise. Two methods of constructing such fractions are considered, namely, based on a chain of equations for the combined moments or a chain of equations for the combined cumulants of the solution and a random force.

As WE KNOW, dynamic systems subjected to the action of random forces, are usually described using the apparatus of Markov processes and diffusion-type processes (see, for example, [1, 2]). However, analytic solutions of the corresponding Fokker–Planck equations have only been found in individual cases. The difficulties involved in solving these equations increase considerably if the random actions cannot be approximated by delta-correlated noise.

One can use matrix continued fractions to find the stationary values of the moments of the solution of such systems. The application of matrix continued fractions to non-linear systems with delta-correlated forces was considered in [3, 4]. The basic principle of this method consists of introducing vectors, whose components are moments of the solution. In a number of cases, the linking of these vectors, due to the non-linearity, takes the form of a three-term interaction, and the corresponding solutions of the required (lowest) moments take the form of matrix continued fractions. For systems with non-white Gaussian noise, the components of the vectors may be combined moments or combined cumulants of the solution and the random force. Below we consider both of these representations and, using a specific example, we demonstrate the advantages of the approach using combined cumulants. The use of this method to find moments of more complex stochastic systems does not involve any difficulties in principle, apart from the increase in the dimensions of the corresponding matrices.

1. Consider the dynamic system described by the following stochastic equation:

$$Tx' + x + \beta x^3 = \xi(t) \tag{1.1}$$

where $\xi(t)$ is a Gaussian Markov process, which can be defined by the additional equation

$$\Pi^{-1}\xi' + \xi = \eta(t) \tag{1.2}$$

T is the time constant of the corresponding linear system, Π is the width of the spectrum of the noise ξ, and η is a Gaussian delta-correlated process with zero mean. Using (1.1) and (1.2) we can obtain in a standard way a system of linked equations for the combined moments or the combined cumulants

$$\langle nm\rangle \equiv \langle x^n\xi^m\rangle, \qquad \langle n, m\rangle \equiv \langle x^n, \xi^{[m]}\rangle$$

The last average is a cumulant bracket of the order of $(m+1)$ of the process $y = x^n$ and the random force ξ, which occurs m times; we have used the notation employed in [5]. The stationary values of the combined moments are related by the equation

$$(n\tau + m)\langle nm\rangle - n\tau\langle (n-1)(m+1)\rangle + n\beta\tau\langle (n+2)m\rangle = m(m-1)D\langle n(m-2)\rangle \tag{1.3}$$
$$m, n = 0, 1, \ldots$$

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and the values of the combined cumulants are related by the equation

$$(n\tau + m)\langle n, m\rangle - n\tau < (n-1),(m+1)\rangle + n\beta\tau < (n+2), m\rangle = mn\langle x^2\rangle_0 < (n-1),(m-1)\rangle \tag{1.4}$$

Here

$$D = \langle \xi^2 \rangle, \quad \tau = \Pi T, \quad \langle x^2\rangle_0 = D\tau \tag{1.5}$$

The parameters D and τ are the intensity and relative correlation time of the input noise, respectively, and $\langle x^2\rangle_0$ is the mean intensity of the output coordinate of the linear ($\beta = 0$) system.

By an appropriate choice of the vectors X_n, both relation (1.3) and (1.4) lead to chains of three-term interaction

$$A_n X_n + B_n X_{n+1} = C_n X_{n-1}, \quad n = 1, 2, \ldots \tag{1.6}$$

where A_n, B_n and C_n are certain matrices. The solution of these chains can be represented by a matrix continued fraction

$$X_n = \cfrac{C_n X_{n+1}}{A_n + \cfrac{B_n C_{n+1}}{A_{n+1} + \cfrac{B_{n+1} C_{n+2}}{A_{n+2} + \ldots}}}, \quad n = 1, 2, \ldots \tag{1.7}$$

(for more details on the realization of the appropriate computational algorithm see [3]). Note that in the case of delta-correlated noise $\xi(\tau \to 0, \langle x^2\rangle_0 = \text{const})$, the mean intensity can be represented by the one-dimensional continued fraction

$$\langle x^2\rangle = \cfrac{\langle x^2\rangle_0}{1 + \cfrac{3\sigma}{1 + \cfrac{5\sigma}{1 + \ldots}}}, \quad \sigma = \beta\langle x^2\rangle_0 \tag{1.8}$$

which converges for any values of the parameter σ (although the rate of convergence falls as σ increases).

2. We will define the vectors X_n as follows:

$$X_n = (\langle x^{2n}\rangle, \langle x^{2n-1}\xi\rangle, \ldots, \langle \xi^{2n}\rangle), \quad n = 1, 2, \ldots \tag{2.1}$$

Note that, in view of the Gaussian form of the input noise $\langle \xi^{2n}\rangle = (2n-1)!!D^n$, the last component in (2.1) appears to be superfluous. However, its introduction is necessary so that the linking of the corresponding vectors has a three-term form. The matrices in (1.6) are the basis of (1.3) and (1.2); for the first matrix equation the sum of the indices in (1.3) equals three, for the second it equals five, etc. The non-zero matrix elements are such that

$$\| A_n \|_{i,i} = (N-i)\tau + i - 1, \quad \| A_n \|_{i,i+1} = (i-N)\tau \tag{2.2}$$

$$\| B_n \|_{i,i} = \beta\tau(N-i), \quad \| C_n \|_{i,i-2} = \langle x^2\rangle_0 (i-1)(i-2), \quad N = 2n+1, \quad i = \overline{1, N}, \quad n = 1, 2, \ldots$$

where N are the dimensions of the square matrices A_n (by adding or subtracting zero elements, the matrices B_n and C_n can also be made square).

The procedure for constructing approximate solutions for the moments using (1.7), realized on a computer, is as follows: by putting the vector X_k identically equal to zero, we thereby limit the fraction and we calculate it from bottom to top. The corresponding result for the vector X_1 will be called the kth approximation. If the norm of the difference between adjacent approximations is less than the specified accuracy, the calculations are stopped.

A numerical analysis showed that the procedure based on continued fractions in the combined moments only converges in the region of fairly small values of σ and τ (Fig. 1, curve 1), and hence, cannot be recommended for analysing stochastic systems with intense noise.

3. For stochastic systems with Gaussian noise ξ, there are reasons for assuming that the procedure based on the construction of chains of the combined cumulants $\langle n, m\rangle$ of the set $\{x^n, \xi\}$ is more adequate. Indeed, as m increases the number of Gaussian terms in the corresponding cumulant bracket increases, which obviously leads to a reduction in its relative value.

We will introduce the vectors X_n, whose components, apart from the initial moments, will be combined cumulants

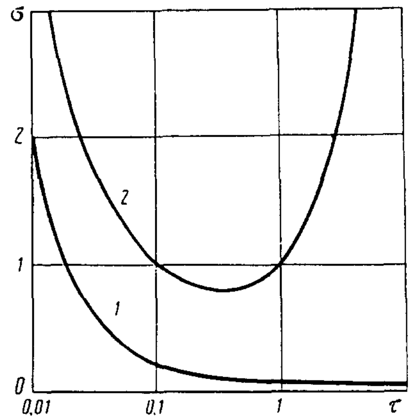

FIG. 1.

$$X_n = (\langle x^{2n}\rangle, \langle \xi, x^{2n-1}\rangle, \dots, \langle \xi^{[n-1]} x\rangle), \quad n = 1, 2, \dots \tag{3.1}$$

(note that their dimensions are now equal to $2n$, and a "superfluous" component is not required). Using (1.4) and (3.1) we find the non-zero elements of the matrices

$$\begin{gathered} \| A_n \|_{i,i} = (2n - i + 1)\tau + i - 1, \quad \| A_n \|_{i,i+1} = (i - 2n - 1)\tau \\ \| B_n \|_{i,i} = (2n - i + 1)\beta\tau, \quad \| C_n \|_{i,i-1} = \langle x^2 \rangle_0 (i - 1)(2n - i + 1) \\ i = \overline{1, 2n}, \quad n = 1, 2, \dots \end{gathered} \tag{3.2}$$

A numerical analysis, carried out using (1.7) and (3.2), showed that the procedure for finding the moments, carried out by an expansion in continued fractions in terms of the vectors (3.1), converges in the region of the parameters σ and τ which lie below curve 2 in Fig. 1. This region is much wider than the corresponding expansion in moment vectors (2.1). Note that the convergence of the method now also occurs in the range of long correlation times. This enables us to compare the corresponding results obtained for $\tau \gg 1$ ($D = \text{const}$) with the accurate quasi-static values of the moments of the process x, which can be obtained by integrating the probability density

$$W_\infty(x) = \frac{1 + 3\beta x^2}{\sqrt{2\pi D}} \exp\left[-\frac{x^2(1 + \beta x^2)^2}{2D}\right] \tag{3.3}$$

[the latter is found in an elementary manner from the quasi-static relation between x and ξ, obtained by omitting the derivative in the initial equation (1.1)].

Figure 2 shows the mean intensity as a function of the relative correlation time of the random force. Curves 1 and 2, drawn from $\langle x^2\rangle_0 = \text{const}$, enable us to pass to the limit of the known results for white noise ($\tau \to 0$). Curves 3 and 4, drawn for $D = \text{const}$, enable us to compare the results obtained with the quasi-static results

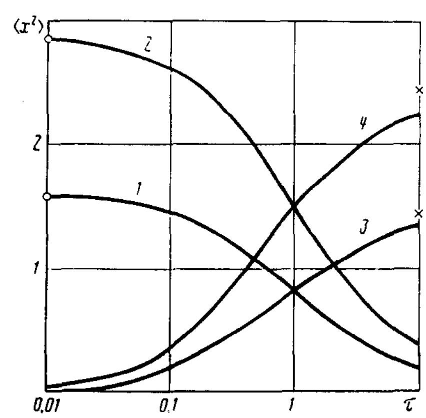

FIG. 2.

($\tau \to \infty$). The white-noise values of the intensity are shown by the small circles on the line $\tau = 0.01$, while the crosses on the line $\tau = 10$ represent the quasi-static values.

For curves 1 and 2, $\langle x^2 \rangle_0 = 1$ and 2, respectively. For curves 3 and 4, $D = 1$ and 2, respectively. Everywhere $\beta = 0.1$.

Thus, using the example of a one-dimensional system with a cubic non-linearity, we have established that the method which uses an expansion in combined cumulants of the solution and of the random force is preferable. There is reason to believe that this conclusion also holds for more-complex systems with additive random action. Of course, as the order of the system increases, as when several power terms of the expansion of the non-linearity are taken into account [4], the dimensions of the corresponding matrices increase, but these difficulties are purely technical. The question of the convergence of this procedure is of interest and remains open.

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